Magic Squares: Superstitions, Mystery and the Plague

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A magic square is a combination of numbers arranged in a square such that the sum of each row, column and both the corner diagonals is the same. This sum is termed the summation (S) of any given magic square.

Magic squares have existed since prehistoric times. Examples of magic squares have been found in ancient Chinese and Indian literature. As early as the eighth century, Arabian astrologers extensively utilized magic squares in astronomy, using magic squares to construct horoscopes. It is likely that magic squares were so called because the properties they possessed seemed to be otherworldly, extraordinary and wonderful. German scholar Cornelius Agrippa even devised magic squares of orders between 3 and 9 and associated them with the celestial bodies Mercury, Venus, Mars, Saturn, Jupiter, the Sun and the Moon. In fact, magic squares were so superstitiously revered that many people engraved them on silver vessels and considered these vessels as "good luck charms" against the plague. Although magic squares lack any immediate practical applications, their position in the world of recreational mathematics has been evergreen.

Characteristics of Magic Squares

There are many different types of magic squares. Magic squares of odd and even orders are governed by different rules of construction, so we'll study them separately. This is obvious, in that the starting step in the construction of odd magic squares usually involves the center box. Even magic squares do not have center boxes, so we employ a different tactic.

There exist 880 squares of order four, manually counted by Frénicle de Bessy in 1693. The number of 5×5 magic squares was similarly found to be 275305224. And the number of 6×6 squares is not known, because there is no clear-cut method to estimate the number of magic squares of any arbitrary order n. However, scientists have used Monte Carlo simulation methods (intensive random sampling of magic squares) to approximate the number of 6×6 squares to $1.774500016 \times 10^{19}$.

There exists a magic sum, or magic number, defined above as the summation S of each magic square. For a magic square of any order the magic sum is

$$S = \frac{1}{n}\sum_{k=1}^{n^2} k$$

$$\frac{1}{2}n(n^2+1)$$

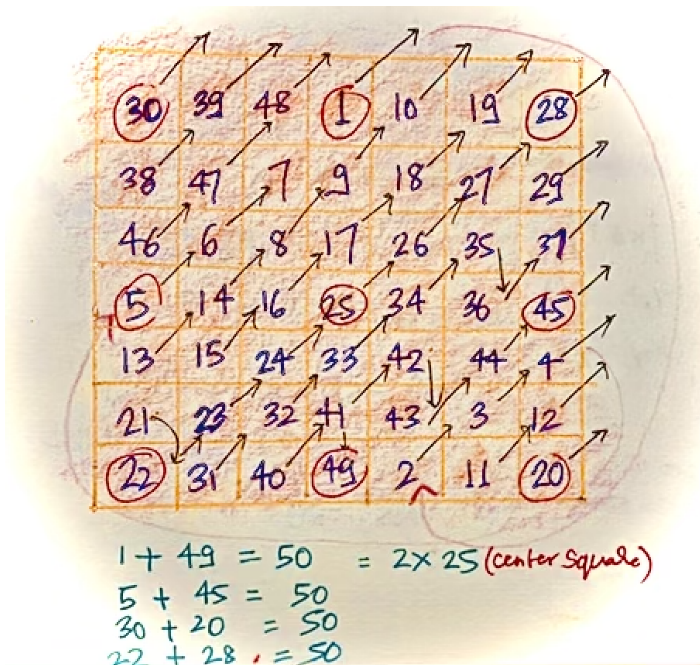
Magic Squares of Odd Orders

A magic square of order 1, with the single entry being 1, is obviously a magic square. Let us consider the next smallest odd magic square, the 3×3 square. There exists only one magic square of order 3 and it has been known since prehistoric times. It was so revered that prehistoric Chinese mathematicians even had a special name for it- Lo Shu.

Knowing all the properties of magic squares, let's look at some methods to construct them.

The simplest method to construct odd order magic squares is the de la Loubere method or the Siamese method. We start by placing 1 in the central box of the top row. Now we can fill the boxes in increments. Place following numbers 2, 3, 4... etc in the square to the top right of the previous square. If no such square exists, we follow a "wrapping around" method. If no square exists to the top or to the right, we respectively return to the bottom and ti the left, in that order of priority. If this square (s) is/are filled we place the number below the previous one.

Illustrating on a 7×7 square,



Can you figure out another method to construct this square? (Hint: Look for patterns in the positions of the odd and even numbers and the shape their boxes make)

Magic Squares of Even Orders

It is evident that we cannot construct a magic square of order two. There is no way that the sum of the diagonals, rows and columns are the same unless all four boxes contain the same number.

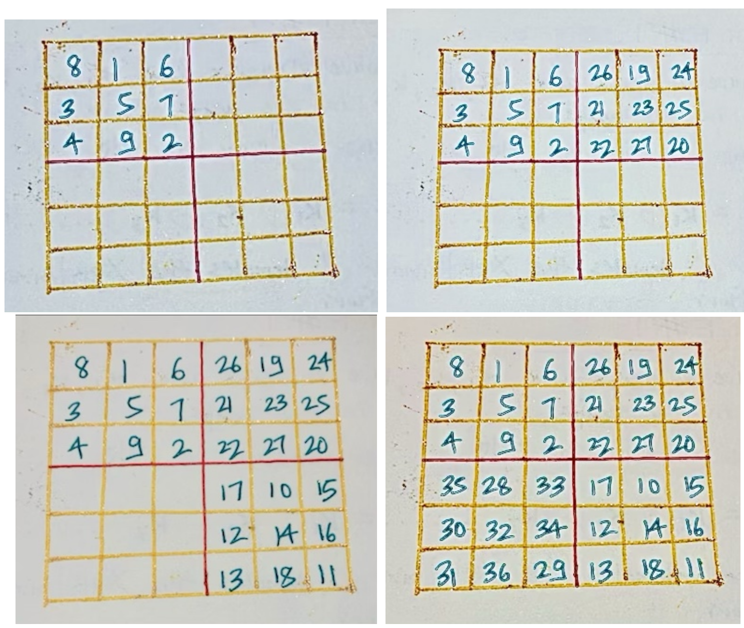
We segregate our discussion of even order magic squares into singly and doubly even squares. Singly even magic squares are of orders $4n+2, n$ being an integer while doubly even magic squares are of orders $4n$.

Singly Even Magic Squares

This is perhaps the most complicated type of magic square. The method we'll use to generate the smallest singly even magic square (of order 6) is quite intriguing.

What is the magic number for this square? It's 111! Recall that the magic number for a magic square of order n is $n(n^2 + 1)$.

Combinatorial problem solving often involves the process of dividing and conquering, that is simplifying the problem and then generalizing. Let's first partition the 6×6 square into four 3×3 squares. In the first block we'll simply enter numbers 1 to 9 using the Siamese method. This is pretty straightforward. Repeat for the lower right square, the upper right square and then the lower left square (diametrically opposite squares).



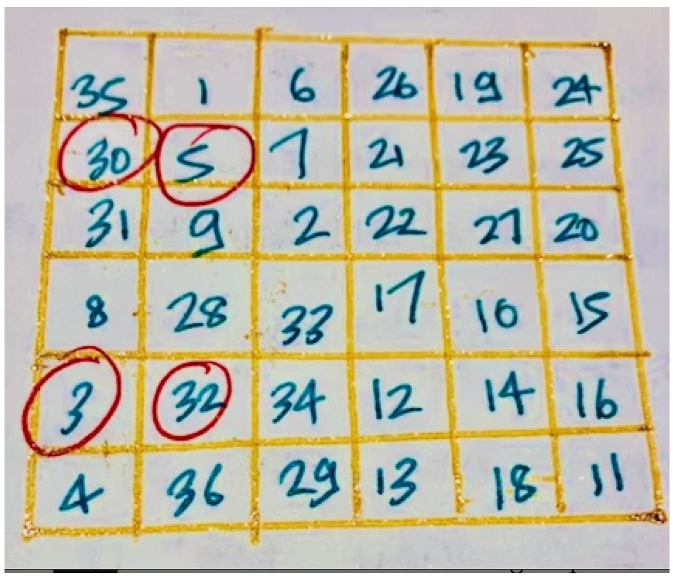
All the columns sum to 111! We're headed in the right direction. But there's a problem: the rows do not sum to 111. The first three and last three rows respectively sum to 84 and 138.

What can we do now?

We do not want to interchange numbers in different columns because they already sum to 111. We've found our first hint: we should ideally only interchange numbers in the same column.

Let's look at the first three rows first. 111-84=27. We need to increase the sum in each column by 27. The next three rows need to be decreased by 138-111=27! We have our solution! We can exchange entries in the first 3 rows with entries in the last 3. 8 in the first row and 35 in the fourth row differ by 27, let's exchange them. Proceed similarly for 3 in the second row with 30 in row five and 4 in the third row with 31 in the last row.

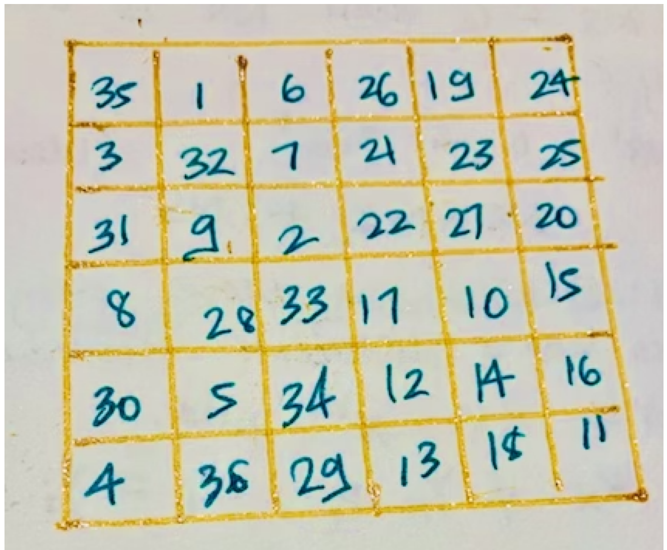
The resultant magic square should look like this:



We've made progress! Now both the rows and the columns all sum to 111!

There's just one slight problem.. the diagonals do not sum to 111. And the process of correcting this glitch doesn't seem obvious. And it's not- until you observe carefully. We need to now move elements around without changing the row or column sums. On experimenting you will find that the required magic square can be obtained by interchanging 3 and 30; and 5 and 32.

We need to switch two elements that are on the diagonals and in the same column. We see that if we return the 3 and 30 to their original places and in column two exchange the 5 and 32, which are diagonal elements, we have not changed the row sums or column sums, but we have changed the diagonals to now sum to 111 each.



And we have our 6×6 magic square!

Can you try constructing a magic square of order 10×10?

Can we now generalize this method for all singly even magic squares of orders $4n+2$? Of course we begin by dividing the square into four $(2n+1)\times(2n+1)$ squares.

Begin with the top left square and place numbers 1 to $\frac{n^2}{4}$. Then fill the centrally opposite bottom right square with numbers $\frac{n^2}{4}+1$, $\frac{n^2}{2}+4$, ... n^2 using the Siamese method. Numbers till $\frac{3n^2}{4}$ go in the upper right square and numbers from $\frac{3n^2}{4}+1$ to n^2 go in the bottom left.

To rectify the row summations exchange corresponding numbers in the upper and lower squares in the first $(n-1)$ columns and the last $(n-1)$ columns. In the n^{th} column exchange all numbers but the central one.

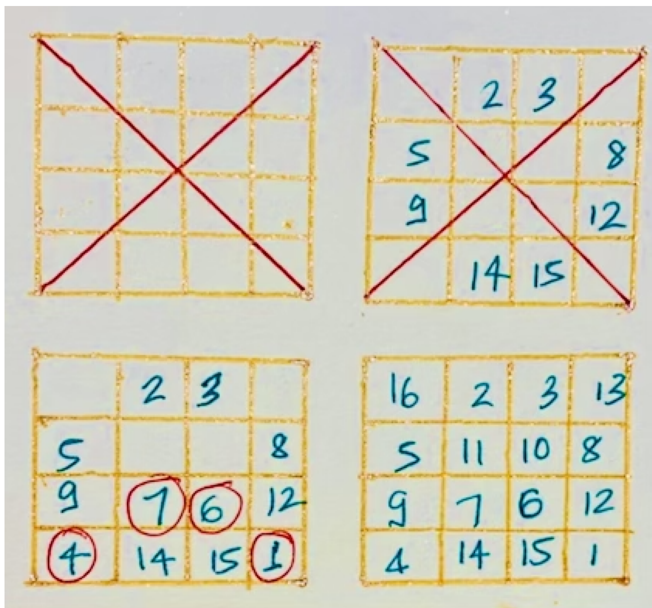
To rectify diagonal summations, the final step, exchange the numbers in the upper and lower squares of the $(k+1)^{st}$ column.

And you should have your k×k magic square! Use this method to construct a 10×10 magic square and check if the last one matches!

Doubly Even Magic Squares

Doubly even magic squares are of orders 4n, n being a natural number. The process of constructing a doubly even magic square is much simpler than the last type, so don't worry!

Let's start with the simplest one: the fourth order magic square. Here is one of the easiest methods of construct a fourth order magic square:



Can you guess the pattern?

We draw the two diagonals and start at the top left corner. We start with 1 and proceed rightward, row by row, only adding numbers in the blank boxes and skipping the numbers in the crossed boxes.

Once we're done, we reverse the process. Now we start with 1 on the bottom right and proceed leftward, row by row, bottom to top, and end with 16 on the top left square.

A final exercise: Can you use this information to construct an 8×8 magic square? (Hint: Partition into four squares, follow the doubly even method and generalize!)